Pythagorean Multi-Proof Square

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Abstract

This paper presents a novel figure for teaching multiple geometric proofs of the Pythagorean theorem. Because it consists only of congruent given right triangles, the figure can be constructed using a template of the given right triangle or, if available, a computer program. Within the figure, called a Pythagorean multi-proof square, there are four a-squares, four b-squares, and one c-square, which in various combinations form the substrate for 12 different Pythagorean proofs. A set of teaching slides is provided in the Appendix.

Pythagorean Multi-Proof Square

Introduction

For nearly a century, mathematics teachers in need of a reference work on geometric proofs of the Pythagorean theorem have used the compendium of 255 geometric proofs by dissection published in a book by Elisha Loomis, which was reprinted by the National Council of Teachers of Mathematics in their journal, *Mathematics Teacher* [1] (full text is available in the ERIC database). In his book, Loomis sub-categorized the geometric proofs according to the geometric arrangements of a-square, b-square, and c-square in the proofs' figures and gave a single proof for a single figure. The current paper makes available to mathematics teachers a new tool for teaching proofs of the Pythagorean theorem, a Pythagorean multi-proof square (Figure 1), which enables derivation of a dozen different Pythagorean geometric proofs by dissection of a single figure.

Methods and Results

To construct a Pythagorean multi-proof square (Figure 1), use either a computer program, such as PowerPoint, or a hand-held template of the given right triangle, as it is constructed entirely of congruent given right triangles (see the Basic Teaching Slides in the Appendix). The procedure is to add to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL. Each newly added triangle is secured in its place by coinciding with at least one vertex in, and overlapping at least one line segment of, the polygon to which it is added. The first eight of the 16 congruent given right triangles

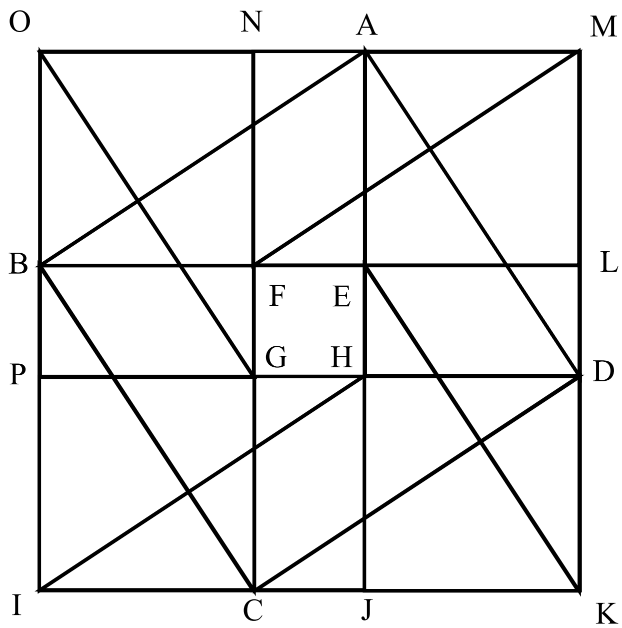


Figure 1. Pythagorean multi-proof square

serve to construct the c-square (square ABCD) and the figure's perimeter square (square MOIK), while the next eight congruent given right triangles serve to complete the construction of the four a-squares (squares DHKJ, AELM, BFNO, and CGPI) and the four b-squares (squares CKLF, DMNG, AOPH and BIJE), so that every side of the four a-squares, four b-squares, and one c-square is coincident with a side of a given right triangle. Every angle of every equilateral quadrilateral in the Pythagorean multi-proof square is a right angle by one of the following justifications: 1) by being congruent with the right angle of a given right triangle; 2) by being comprised of adjacent complementary angles of given right triangles; or 3) by being adjacent to a right angle on a straight line.

The Pythagorean multi-proof square contains 12 different types of geometric arrangements, selected from its four a-squares, four b-squares, and one c-square, that can be used to prove the Pythagorean theorem (Figure 2). In the 12 maps shown in

Figure 2, the notations “a”, “b”, and “c” denote a-square, b-square, and c-square, respectively; the notation “a,b” denotes a-square encompassed by b-square; and the notation “n” denotes the number of times the particular geometric arrangement is replicated by a different combination of the four a-squares, four b-squares, and one c-square upon rotation of the figure. For example, the combinations that make up the four replicates of the Type 1 geometric arrangement are as follows: 1) Squares CGPI, CKLF, and ABCD; 2) Squares DHJK, DMNG, and ABCD; 3) Squares AELM, AOPH, and ABCD; and 4) Squares BFNO, BIJE, and ABCD. Altogether, the replicates of all the geometric arrangements (Types 1-12) add up to a total of 44 different combinations of the four a-squares, four b-squares, and one c-square that can be used to prove the Pythagorean theorem.

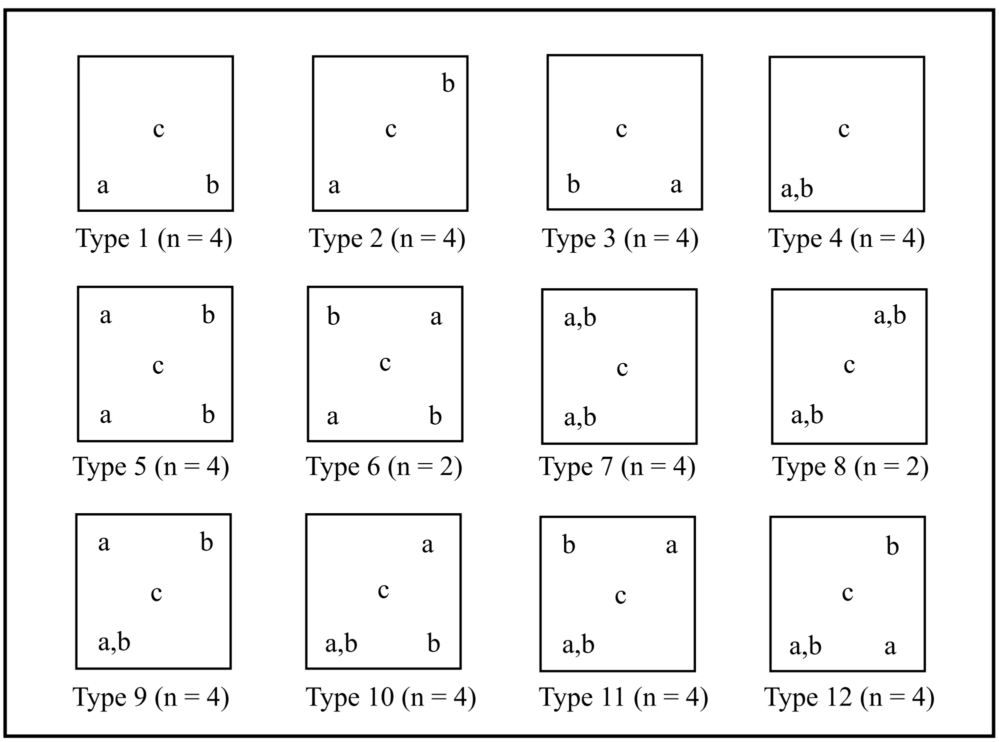


Figure 2. Maps of geometric arrangements for proving the Pythagorean theorem

Proofs From the Geometric Arrangements

Only Types 1-4 of the 12 different geometric arrangements in the Pythagorean multi-proof square have been used previously to prove the Pythagorean theorem. Thabit ibn Qurra used Types 2 and 3 in the 9th century A.D. [2]. Incidentally, it is of historical interest that the Pythagorean multi-proof square displays the entire figures that he used in his two proofs! Loomis used Types 1 and 4 in his figures for Proofs 165 and 131, respectively, early in the last century [1].

Herein, the Pythagorean proofs for Types 1-4 are based on the equalities, Area (MOIK) = Area (a-square) + Area (b-square) + 4Area (given right triangle) = Area (c-square) + 4Area (given right triangle), while the proofs for Types 5-12 are based on the equalities, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) – Area (EFGH). Written proofs for all 12 different types of geometric arrangements in the Pythagorean multi-proof square follow below:

Type 1 Pythagorean proof (Squares CGPI, CKLF, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 1 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG); and Area (c-square) = Area (ABCD).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (c-square) + 4Area (given right triangle).

Area (MOIK) = Area (CGPI) + Area (CKLF) + Area (GOP) + Area (GON) + Area (FMN) + Area (FML); but, since congruent given right triangles, ADM, CDK, BCI, ABO, GON, GOP, FMN and FML have the same areas, Area (MOIK) = Area (CGPI) + Area (CKLF) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (a-square) + Area (b-square) + Area 4(Area (given right triangle).

Subtracting equations for Area (MOIK) and rearranging gives Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 2 Pythagorean proof (Squares CGPI, DMNG, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 2 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG); and Area (c-square) = Area (ABCD).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (c-square) + 4Area (given right triangle).

Area (MOIK) = Area (CGPI) + Area (DMNG) + Area (CDK) + Area (CDG) + Area (GOP) + Area (GON); but, since congruent given right triangles, ADM, CDK, BCI, ABO, GON, GOP, FMN and FML have the same areas, Area (MOIK) = Area (CGPI) + Area (DMNG) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (a-square) + Area (b-square) + 4Area (given right triangle).

Subtracting equations for Area (MOIK) and rearranging gives Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 3 Pythagorean proof (Squares DHJK, BIJE, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 3 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG); and Area (c-square) = Area (ABCD).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (c-square) + 4Area (given right triangle).

Area (MOIK) = Area (DHJK) + Area (BIJE) + Area (ABO) + Area (ABE) + Area (ADM) + Area (ADH); but, since congruent given right triangles, ADM, CDK, BCI, ABO, GON, GOP, FMN and FML have the same areas, Area (MOIK) = Area (DHJK) + Area (BIJE) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (a-square) + Area (b-square) + 4(Area (given right triangle).

Subtracting equations for Area (MOIK) and rearranging gives Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 4 Pythagorean proof (Squares CGPI, BIJE, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 4 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG); and Area (c-square) = Area (ABCD).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (c-square) + 4Area (given right triangle).

Area (MOIK) = Area (CGPI) + Area (DMNG) + Area (GOP) + Area (GON) + Area (CDK) + Area (CDG); but, substituting equivalent Area (BIJE) for Area (DMNG) and substituting congruent given right triangles ADM, CDK, BCI, and ABO for congruent given right triangles GOP, GON, CDK, and CDG gives Area (MOIK) = Area (CGPI) + Area (BIJE) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO) = Area (a-square) + Area (b-square) + 4Area (given right triangle).

Subtracting equations for Area (MOIK) and rearranging gives Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 5 Pythagorean proof (Squares BFNO, CGPI, CKLF, DMNG, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 5 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right

triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (DMNFEH) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (BFNO) + Area (BIJE) + Area (DHJK) + Area (DMNFEH); substituting equivalent areas, Area (CGPI), Area (CKLF), and (Area (DMNG)-Area (EFGH)), for Area (DHJK), Area (BIJE), and Area (DMNFEH), respectively,
Area (MOIK) = Area (BFNO) + Area (CGPI) + Area (CKLF) + Area (DMNG) - Area (EFGH), then, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives, 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 6 Pythagorean proof (Squares CGPI, AELM, CKLF, AOPH, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 6 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (DMNG) = Area (BIJE) = Area (CKLF) = Area (AOPH) = Area (AOPGFE) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (CGPI) + Area (AELM) + Area (CKLF) + Area (AOPGFE);

substituting an equivalent area, (Area (AOPH) – Area (EFGH)), for Area (AOPGFE), Area (MOIK) = Area (CGPI) + Area (AELM) + Area (CKLF) + Area (AOPH) - Area (EFGH). Then, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 7 Pythagorean proof (Squares BFNO, CGPI, AOPH, BIJE, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 7 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (AOPGFE) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI,

and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (CGPI) + Area AELM) + Area (CKLF) + Area (AOPGFE); and substituting equivalent areas, Area (BFNO), Area (BIJE), and (Area (AOPH) – Area (EFGH)), for Area (AELM), Area (CKLF), and Area (AOPGFE), respectively, Area (MOIK) = Area (CGPI) + Area (BFNO) + Area (BIJE) + Area (AOPH) - Area (EFGH), then, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 8 Pythagorean proof (Squares CGPI, AELM, BIJE, DMNG, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 8 Proof).   Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral

quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (DMNFEH) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (BFNO) + Area (BIJE) + Area (DHJK) + Area (DMNFEH); and substituting equivalent areas, Area (CGPI), Area (AELM), and (Area (DMNG) - (Area (EFGH)) for Area (DHJK), Area (BFNO), and Area (DMNFEH), respectively, Area (MOIK) = Area (CGPI) + Area (AELM) + Area (BIJE) + Area (DMNG) - Area (EFGH), then, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 9 Pythagorean proof (Squares BFNO, CGPI, BIJE, DMNG, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 9 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (DMNFEH) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (BFNO) + Area (BIJE) + Area (DHJK) + Area (DMNFEH); and substituting equivalent areas, Area (CGPI) and (Area (DMNG) - Area (EFGH)) for Area (DHJK) and Area (DMNFEH), respectively, Area (MOIK) = Area (CPGI) + Area (BFNO) + Area (BIJE) + Area (DMNG) - Area (EFGH), then, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 10 Pythagorean proof (Squares CGPI, AELM, BIJE, CKLF, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 10 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (CKLEHG) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area

(BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (CGPI) + Area (AELM) + Area (AOPH) + Area (CKLEHG); substituting equivalent areas, Area (BIJE) and (Area (CKLF) – Area (EFGH)) for Area (AOPH) and Area (CKLEHG), respectively, then Area (MOIK) = Area (CGPI) + Area (AELM) + Area (CKLF) + Area (BIJE) - Area (EFGH). Thus, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives, 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 11 Pythagorean proof (Squares AELM, CGPI, AOPH, BIJE, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 11 Proof). Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area

(CKLF) = Area (DMNG) = Area (AOPGFE) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (CGPI) + Area (AELM) + Area (CKLF) + Area (AOPGFE); and substituting equivalent areas, Area (BIJE) and (Area (AOPH) – Area (EFGH)), for Area (CKLF) and Area (APOGFE), respectively, then Area (MOIK) = Area (CGPI) + Area (AELM) + Area (BIJE) + Area (AOPH) - Area (EFGH). Thus, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Type 12 Pythagorean proof (Squares CGPI, DHJK, BIJE, DMNG, and ABCD).

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

Proof of Pythagorean Theorem (Type 12 Proof).   Let side a be the short side if the given right triangle is scalene. Use congruent given right triangles to construct a Pythagorean multi-proof square like that shown Figure 1 by adding to given right

triangle ABE 15 congruent given right triangles in the following sequence: BCF, CDG, ADH, ABO, BCI, CDK, ADM, FML, FMN, GON, GOP, HIP, HIJ, EKJ, and EKL.

In Figure 1, since every side of every a-square, b-square, and c-square is coincident with a side of a given right triangle and every angle of every equilateral quadrilateral is a right angle, Area (a-square) = Area (AELM) = Area (BFNO) = Area (CGPI) = Area (DHJK); and Area (b-square) = Area (AOPH) = Area (BIJE) = Area (CKLF) = Area (DMNG) = Area (AOPGFE) + Area (EFGH); and Area (c-square) = Area (ABCD) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG) + Area (EFGH).

Area (MOIK) = Area (ABCD) + Area (ADM) + Area (CDK) + Area (BCI) + Area (ABO); substituting congruent triangles ADH, ABE, BCF and CDG for ADM, CDK, BCI, and ABO, Area (MOIK) = Area (ABCD) + Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG); since, Area (ABCD) - Area (EFGH) = Area (ADH) + Area (ABE) + Area (BCF) + Area (CDG), then, Area (MOIK) = 2Area (ABCD) - Area (EFGH). Thus, Area (MOIK) = 2Area (c-square) – Area (EFGH).

Area (MOIK) = Area (CGPI) + Area (AELM) + Area (CKLF) + Area (AOPGFE); and substituting equivalent areas, Area (DHJK), Area (BIJE), and (Area (DMNG) - Area (EFGH)) for Area (AELM), Area (CKLF), and Area (AOPGFE), respectively, then Area (MOIK) = Area (DHJK) + Area (CGPI) + Area (BIJE) + Area (DMNG) - Area (EFGH). Thus, Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH).

Subtracting equations for Area (MOIK) and rearranging gives 2Area (a-square) + 2Area (b-square) - Area (EFGH) = 2Area (c-square) - Area (EFGH), which simplifies to Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

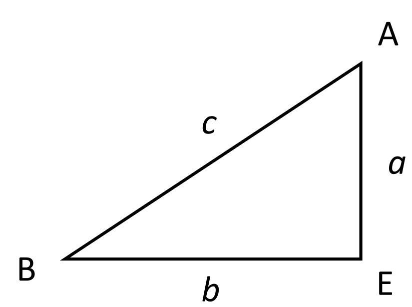
References

1. Loomis, E. *The Pythagorean Proposition,* 1940. Reprinted by National Council of Teachers of Mathematics, Reston, VA, 1968.
2. Shloming, R. *Thabit ibn Qurra and the Pythagorean Theorem*. Mathematics Teacher **63 (1970)**, 519-28.

Appendix

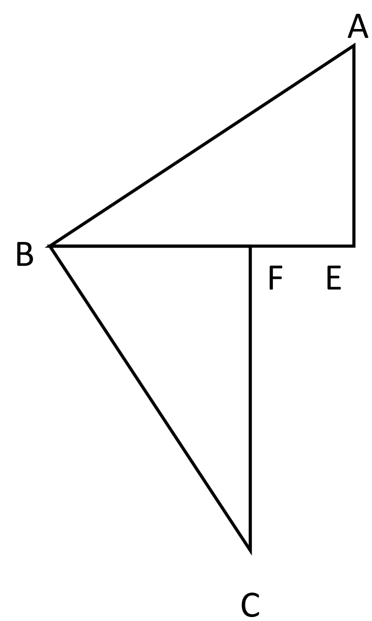
A. Basic Teaching Slides

Pythagorean Theorem. *In any given right triangle with sides a, b, and c, where c is the hypotenuse, Area (a-square) + Area (b-square) = Area (c-square).*

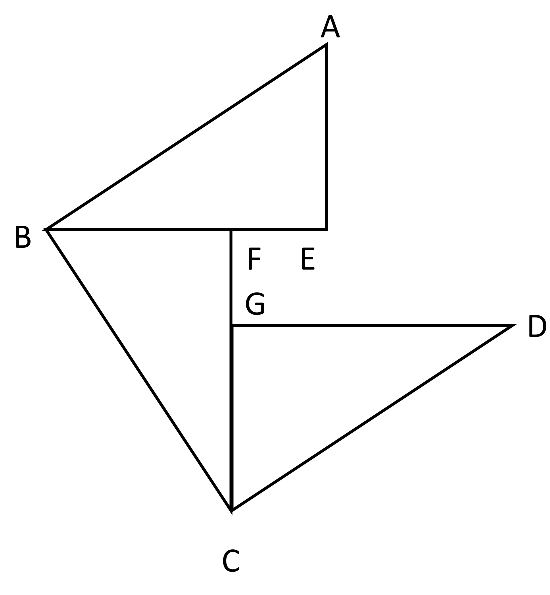


Given right triangle ABE, prove the theorem.

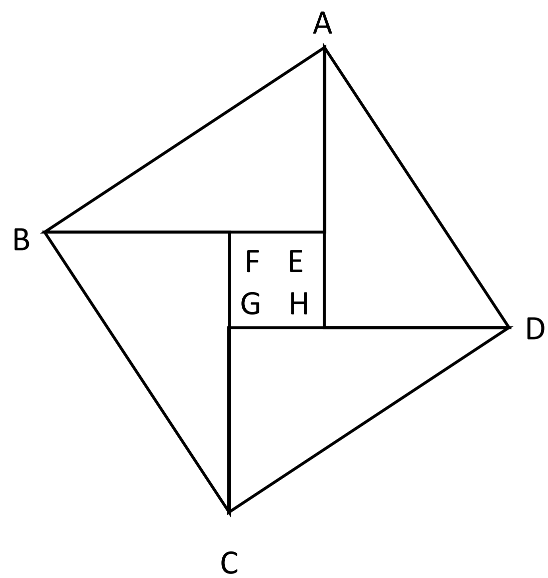
Add congruent given right triangle BCF:



Add congruent given right triangle CDG:

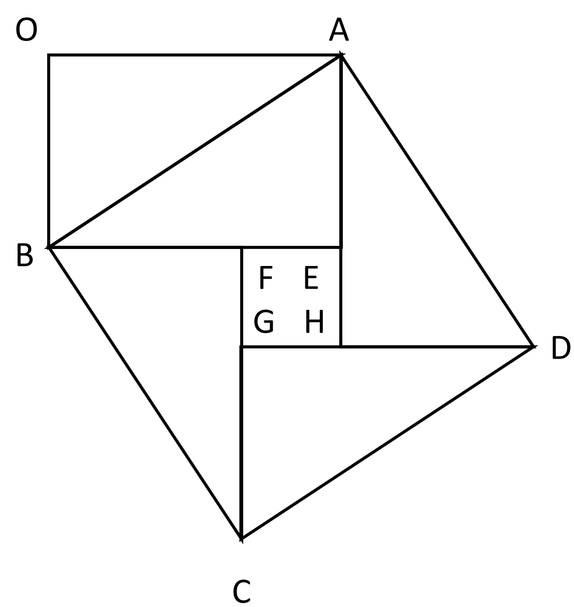


Add congruent given right triangle ADH:

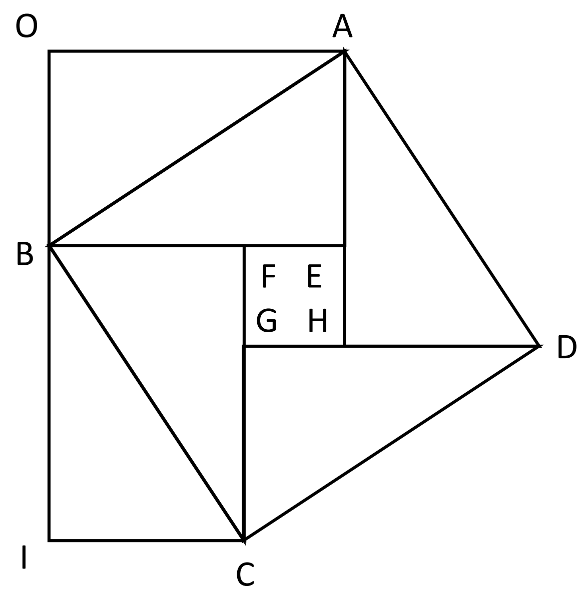


Area (ABCD) = Area (c-square) = 4Area (given right triangle) + Area (EFGH)

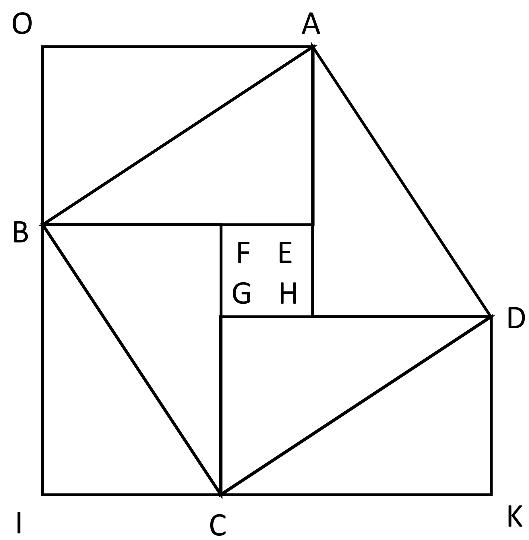
Add congruent given right triangle ABO:



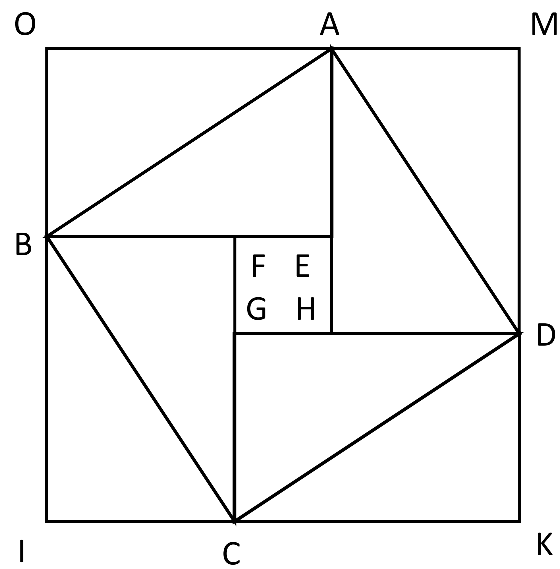
Add congruent given right triangle BCI:



Add congruent given right triangle CDK:

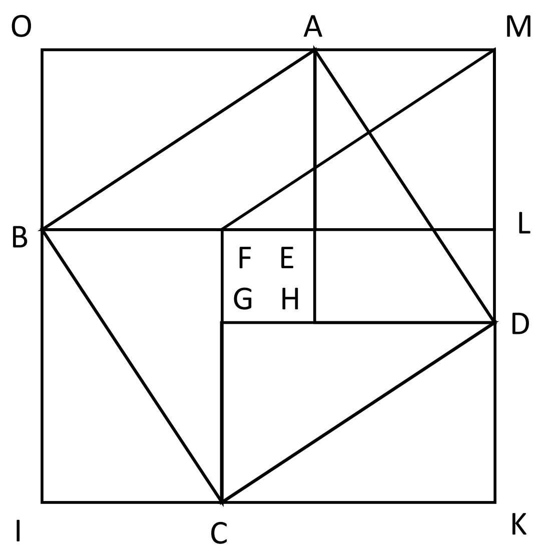


Add congruent given right triangle ADM:

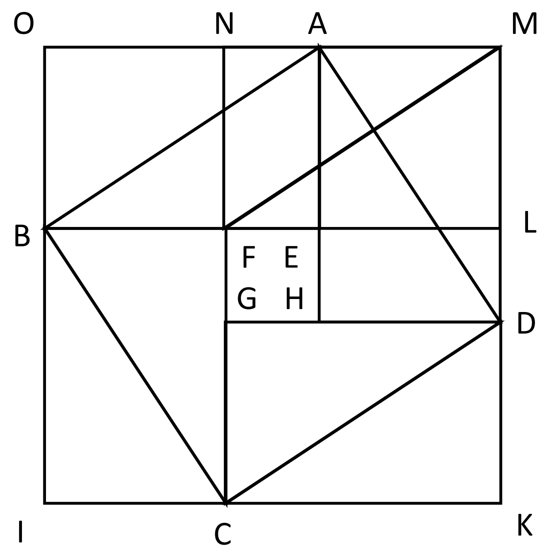


Area (MOIK) = Area (ABCD) + 4Area (given right triangle)
Area MOIK = 2Area (ABCD) - Area (EFGH)
Area (MOIK) = 2Area (c-square) - Area (EFGH)

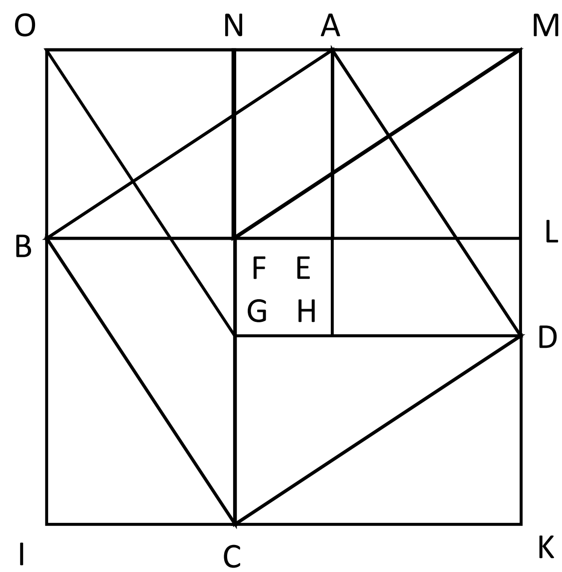
Add congruent given right triangle FML:



Add congruent given right triangle FMN:



Add congruent given right triangle GON:



Area (BFNO) = Area (a-square)
Area (DMNG) = Area (b-square)
Area (MOIK) = Area (BFNO) + Area (DMNG) + 4Area (given right triangle)
Area (MOIK) = Area (a-square) + Area (b-square) + 4Area (given right triangle)

Thus, we have shown that

Area (MOIK) = Area (a-square) + Area (b-square) + 4Area (given right triangle)

And that

Area (MOIK) = Area (c-square) + 4Area (given right triangle).

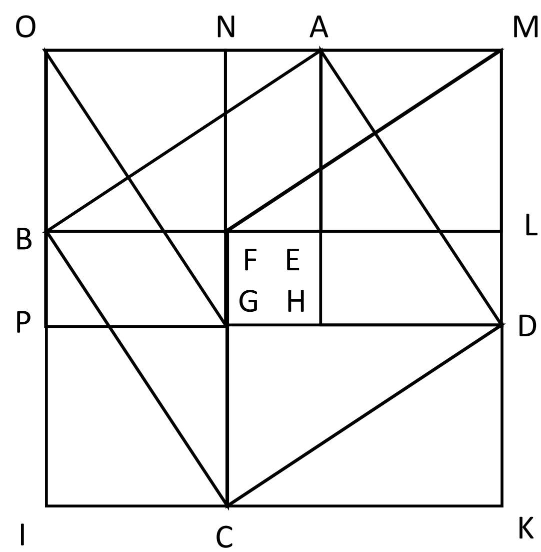
Since two things that equal the same thing equal one another,

Area (a-square) + Area (b-square) + 4Area (given right triangle) = Area (c-square) + 4Area (given right triangle),

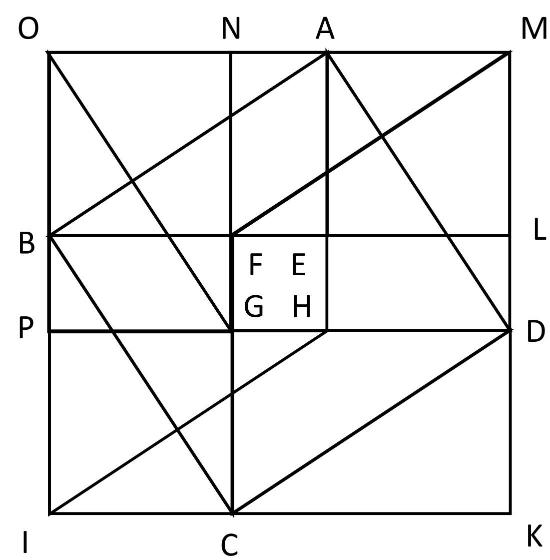
which simplifies to

Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.

Add congruent given right triangle GOP:

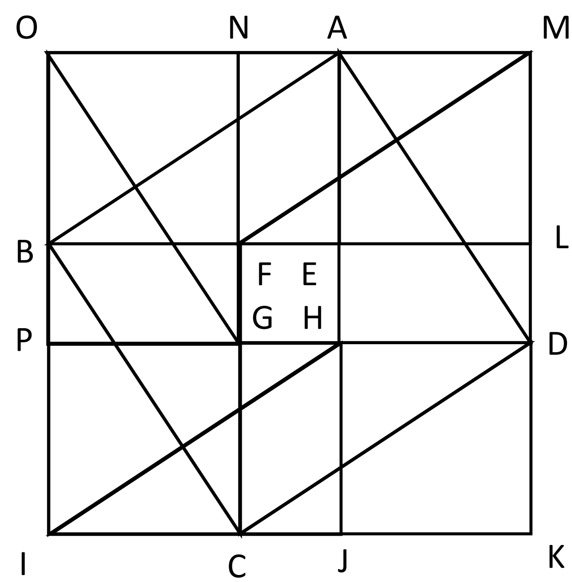


Add congruent given right triangle HIP:

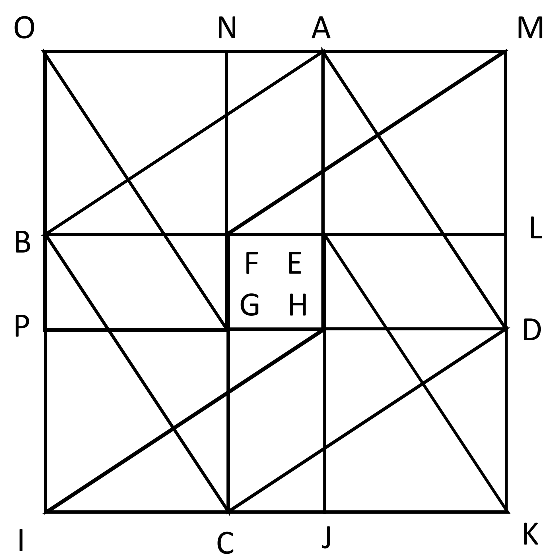


Area (CGPI) = Area (a-square)
Area (AOPH) = Area (b-square)

Add congruent given right triangle HIJ:

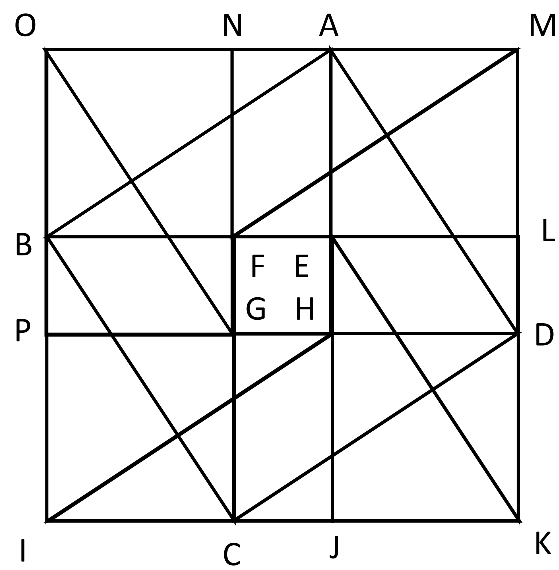


Add congruent given right triangle EKJ:



Area (DHJK) = Area (a-square)
Area (BIJE) = Area (b-square)

Add congruent given right triangle EKL:



Area (AELM) = Area (a-square)
Area (CKLF) = Area (b-square)
Area (MOIK) = Area (AELM) + Area (CKLF) + Area (CPGI) + Area (AOPGFE)
Area (MOIK) = Area (AELM) + Area (CKLF) + Area (CPGI) + Area (AOPH) – Area (EFGH)
Area (MOIK) = 2Area (a-square) + 2Area (b-square) - Area (EFGH)

Thus, we have shown that

Area (MOIK) = 2Area (a-square) + 2Area (b-square) – Area (EFGH)

And that,

Area (MOIK) = 2Area (c-square) – Area (EFGH).

Since two things that equal the same thing equal one another,

2Area (a-square) + 2Area (b-square) – Area (EFGH) = 2Area (c-square) – Area (EFGH),

which simplifies to

Area (a-square) + Area (b-square) = Area (c-square). Q.E.D.